

## Laplace Transforms

Analysis of network can be done using the time domain, frequency domain and the complex frequency domain (s domain). Laplace transforms converts functions in the time domain to s domain using complex variable  $s = \sigma + j\omega$ .

Laplace transformation of a function  $f(t)$  is defined as

$$F(s) = \mathcal{L}f(t) = \int_0^{\infty} f(t) \cdot e^{-st} dt$$

The time function is denoted by lowercase letter and the Laplace transform by capital letter.

Laplace transform exist for  $f(t)$  only when  $t > 0$

Inverse Laplace transforms permits going back in the reverse direction i.e. from s domain to time domain.

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} F(s) e^{st} ds$$

### Basic Functions

#### A Unit step function

The most common driving function in electrical engineering is the unit step function denoted as

$$u(t) = f(x) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases}$$

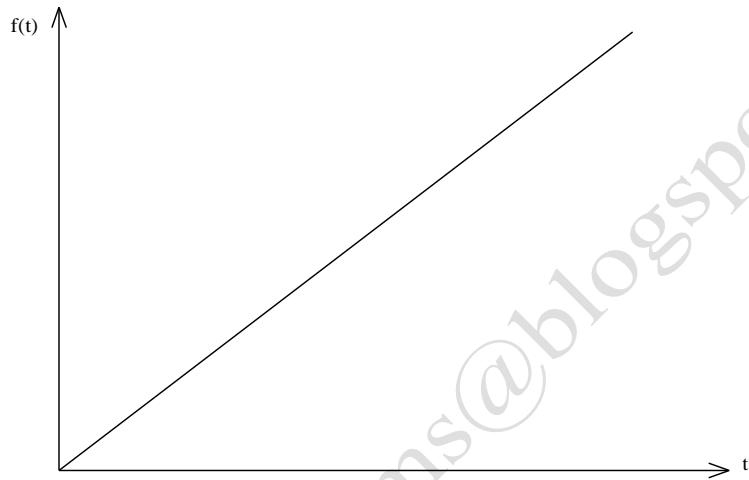


The Laplace transform is

$$\begin{aligned} F(s) &= \int_0^\infty 1 \cdot e^{-st} dt = \left[ -\frac{1}{s} e^{-st} \right]_0^\infty \\ &= 0 + \frac{1}{s} = \frac{1}{s} \end{aligned}$$

## B Ramp Function

$$f(t) = t$$



$$F(s) = \int_0^\infty t \cdot e^{-st} dt$$

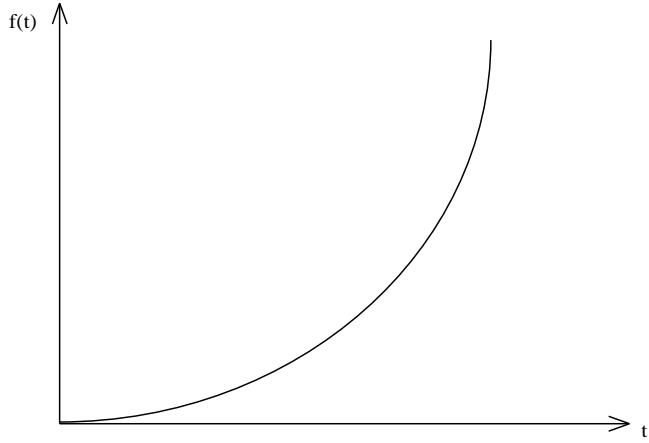
Integrating by parts

$$u = t, \quad \delta v = e^{-st}, \quad \delta u = 1, \quad v = -\frac{e^{-st}}{s}$$

$$\begin{aligned} F(s) &= \left[ t \cdot \left( -\frac{e^{-st}}{s} \right) \right]_0^\infty - \int_0^\infty \left( -\frac{e^{-st}}{s} \right) \cdot 1 \delta t \\ &= 0 - \int_0^\infty -\frac{e^{-st}}{s} \cdot 1 \delta t \\ &= \left[ \frac{1}{s} \left( -\frac{1}{s} e^{-st} \right) \right]_0^\infty = \left[ \left( -\frac{1}{s^2} e^{-st} \right) \right]_0^\infty \\ &= \frac{e^0}{s^2} = \frac{1}{s^2} \end{aligned}$$

## C Parabolic function

$$f(t) = t^2$$



$$F(s) = \int_0^\infty t^2 \cdot e^{-st} dt$$

$$\int uv = uv_1 - u^1 v_2 + u^{11} v_3 - u^{111} v_4$$

$$u = t^2, u^1 = 2t, u^{11} = 2, u^{111} = 0$$

$$v = e^{-st}, v_1 = -\frac{e^{-st}}{s}, v_2 = \frac{e^{-st}}{s^2}, v_3 = -\frac{e^{-st}}{s^3}$$

$$F(s) = \left[ t^2 \left( -\frac{1}{s} e^{-st} \right) \right]_0^\infty - \left[ \left( 2t \cdot \frac{e^{-st}}{s^2} \right) \right]_0^\infty - \left[ \left( 2 \cdot \frac{e^{-st}}{s^3} \right) \right]_0^\infty$$

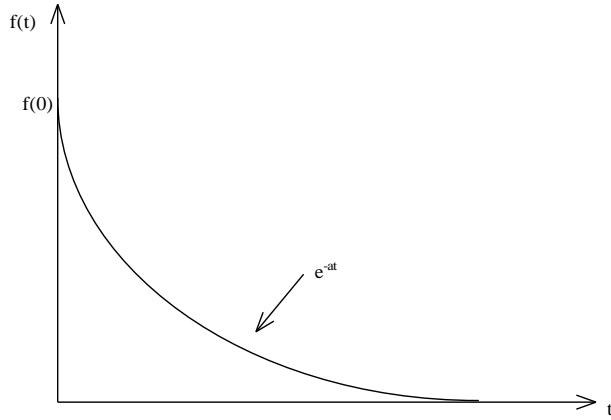
$$F(s) = \frac{2}{s^3}$$

From C and B it shows that these function can be generated using

$$\mathcal{L}f(t) = \mathcal{L} t^n = \frac{n!}{s^{n+1}}$$

## D Exponential function

i.  $f(t) = e^{-\alpha t}$



$$F(s) = \int_0^{\infty} e^{-\alpha t} \cdot e^{-st} dt$$

$$F(s) = \int_0^{\infty} e^{-(s+\alpha)t} dt$$

$$F(s) = \left[ -\frac{1}{s+\alpha} e^{-(s+\alpha)t} \right]_0^{\infty}$$

$$F(s) = \frac{1}{s+\alpha}$$

ii.  $f(t) = t \cdot e^{-\alpha t}$

$$u = t, u^1 = 1, u^{11} = 0$$

$$v = e^{-(s+\alpha)t}, v_1 = -\frac{e^{-(s+\alpha)t}}{s+\alpha}, v_2 = -\frac{e^{-(s+\alpha)t}}{(s+\alpha)^2}$$

$$F(s) = \int_0^{\infty} t \cdot e^{-\alpha t} \cdot e^{-st} dt$$

$$F(s) = \int_0^{\infty} t \cdot e^{-(s+\alpha)t} dt$$

$$F(s) = \left[ t \cdot \left( -\frac{e^{-(s+\alpha)t}}{s+\alpha} \right) \right]_0^{\infty} - \left[ \left( \frac{e^{-(s+\alpha)t}}{s+\alpha^2} \right) \right]_0^{\infty}$$

$$F(s) = \frac{1}{(s+\alpha)^2}$$

From Di and Dii when  $f(t)=t^n e^{-\alpha t}$  then

$$F(s) = \frac{n!}{(s + \alpha)^{n+1}}$$

## E Sinusoidal function

$$f(t) = \sin \omega t$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$F(s) = \frac{1}{2j} \int_0^\infty (e^{j\omega t} - e^{-j\omega t}) e^{-st} dt$$

$$F(s) = \frac{1}{2j} \int_0^\infty e^{-(s-j\omega)t} - e^{-(s+j\omega)t} dt$$

$$F(s) = \frac{1}{2j} \left[ -\frac{e^{-(s-j\omega)t}}{s-j\omega} + \frac{e^{-(s+j\omega)t}}{s+j\omega} \right]_0^\infty$$

$$F(s) = \frac{1}{2j} \left[ \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right]$$

$$F(s) = \frac{1}{2j} \left[ \frac{s+j\omega - s-j\omega}{s^2 + \omega^2} \right] = \frac{1}{2j} \left[ \frac{2j\omega}{s^2 + \omega^2} \right]$$

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

## F Frequency Shifted Sinusoidal Signal

$$f(t) = e^{-\alpha t} \sin \omega t$$

$$F(s) = \frac{1}{2j} \int_0^\infty (e^{j\omega t} - e^{-j\omega t}) e^{-\alpha t} \cdot e^{-st} dt$$

$$F(s) = \frac{1}{2j} \int_0^\infty (e^{j\omega t} - e^{-j\omega t}) e^{-(s+\alpha)t} dt$$

$$F(s) = \frac{1}{2j} \int_0^\infty (e^{-((s+\alpha)-j\omega)t} - e^{-((s+\alpha)+j\omega)t}) dt$$

$$F(s) = \frac{1}{2j} \left[ -\frac{e^{-((s+\alpha)-j\omega)t}}{(s+\alpha)-j\omega} + \frac{e^{-((s+\alpha)+j\omega)t}}{(s+\alpha)+j\omega} \right]_0^\infty$$

$$F(s) = \frac{1}{2j} \left[ \frac{1}{(s+\alpha)-j\omega} - \frac{1}{(s+\alpha)+j\omega} \right]$$

$$F(s) = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

## G Co-sinusoidal function

$$f(t) = \cos \omega t$$

As derived in E and F the Laplace transform of  $\cos \omega t$  is

$$F(s) = \frac{s}{s^2 + \omega^2}$$

And for  $f(t) = e^{-\alpha t} \cos \omega t$

$$F(s) = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

## H Hyperbolic functions

If  $f(t) = \sinh \omega t$ , then  $F(s) = \frac{\omega}{s^2 - \omega^2}$

If  $f(t) = \cosh \omega t$ , then  $F(s) = \frac{s}{s^2 - \omega^2}$

Hint

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$$

$$\sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$$

## First Shifting Theorem

States that if Laplace transform of  $f(t)$  is  $F(s)$  then

$$\mathcal{L}e^{at} \cdot f(t) = F(s - a)$$

## Second Shifting Theorem (Time shifting Theorem)

States that if Laplace transform of  $f(t)$  is  $F(s)$  then Laplace transform of  $f(t - a)$  is

$$\mathcal{L}f(t - a) = e^{-as} \cdot F(s)$$

## Laplace transform of a derivative

Given a function  $f(t)$  it's Laplace transform is

$$\mathcal{L}f(t) = F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$

Integrating by parts

$$u = f(t), \delta v = e^{-st}, \delta u = f'(t), v = -\frac{e^{-st}}{s}$$

$$F(s) = \left[ f(t) \cdot \left( -\frac{e^{-st}}{s} \right) \right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s} \cdot f'(t) dt$$

$$F(s) = \frac{f(0^+)}{s} + \frac{1}{s} \int_0^\infty f'(t) \cdot e^{-st} dt$$

Multiply both sides by s

$$sF(s) = f(0^+) + \mathcal{L}f'(t)$$

$$\mathcal{L}f'(t) = sF(s) - f(0^+)$$

$f(0^+)$  is the initial value of the function at  $t=0$  which is a constant value.

## Laplace transform of an integral

Let the function be  $\int f(t)dt$ , the Laplace transform is

$$\mathcal{L} \int f(t)dt = \int_0^\infty \int f(t)dt \cdot e^{-st} dt$$

Integrating by parts

$$u = \int f(t)dt, \delta v = e^{-st}, \delta u = f(t), v = -\frac{e^{-st}}{s}$$

$$\mathcal{L} \int f(t)dt = \left[ f(t)dt \cdot -\frac{e^{-st}}{s} \right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s} \cdot f(t) dt$$

$$\mathcal{L} \int f(t)dt = \frac{1}{s} \int f(t)dt_{0^+} + \frac{1}{s} \int_0^\infty f(t) \cdot e^{-st} dt$$

$$\mathcal{L} \int f(t)dt = \frac{f(t)dt_{0^+}}{s} + \frac{F(s)}{s}$$

$\int f(t)dt_{0^+}$  gives the value of the integral at  $t=0$

## Initial Value Theorem

The Laplace transform of a differential is

$$\mathcal{L}f'(t) = sF(s) - f(0^+) = \int_0^\infty \frac{\delta}{\delta t} f(t) \cdot e^{-st} dt$$

If  $s$  tends to infinity then

$$\lim_{s \rightarrow \infty} \mathcal{L}f'(t) = \lim_{s \rightarrow \infty} \frac{\delta}{\delta t} f(t) \cdot e^{-st} dt = \lim_{s \rightarrow \infty} sF(s) = f(0^+) = 0$$

$$\lim_{s \rightarrow \infty} sF(s) = f(0^+)$$

Where  $f(0^+)$  is a constant

This expression permits the evaluation of the initial value of the time domain solution of  $f(t)$ , using the transform version.

## Final Value Theorem

Using the Laplace transform of a differential

$$\mathcal{L} \frac{\delta}{\delta t} f(t) = \int_0^\infty \frac{\delta}{\delta t} f(t) \cdot e^{-st} dt = sF(s) - f(0^+)$$

If  $s$  tends to zero then

$$\mathcal{L} \frac{\delta}{\delta t} f(t) = \lim_{s \rightarrow 0} \int_0^\infty \frac{\delta}{\delta t} f(t) dt = \lim_{s \rightarrow 0} sF(s) - f(0^+)$$

$$\mathcal{L} \frac{\delta}{\delta t} f(t) = [f(t)]_0^\infty = \lim_{s \rightarrow 0} sF(s) - f(0^+)$$

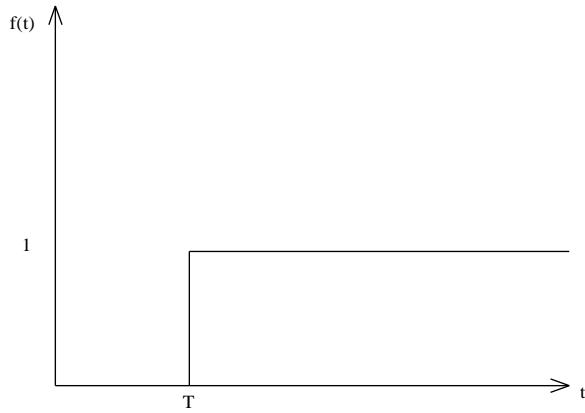
$$\mathcal{L} \frac{\delta}{\delta t} f(t) = f(\infty) - f(0^+) = \lim_{s \rightarrow 0} sF(s) - f(0^+)$$

$$f(\infty) = \lim_{s \rightarrow 0} sF(s)$$

This is useful where the transform solution of the problem is available and the needed information is about the final or steady state solution.

## Time Displacement Theorem

This theorem states that if a function  $f(t)$  is Laplace transformable and



$$\mathcal{L}f(t) = F(s)$$

Then

$$\mathcal{L}f(t - T) = e^{-sT} \cdot F(s)$$

### Example 1.1

Obtain the Laplace transform of  $f(t) = 1 - e^{-at}$  from first principle, a being a constant.

### Solution

$$F(s) = \int_0^{\infty} (1 - e^{-at}) \cdot e^{-st} dt$$

$$F(s) = \int_0^{\infty} e^{-st} - e^{-(s+a)t} dt$$

$$F(s) = \left[ -\frac{e^{-st}}{s} + \frac{e^{-(s+a)t}}{s+a} \right]_0^{\infty}$$

$$F(s) = \frac{1}{s} - \frac{1}{s+a} = \frac{a}{s(s+a)}$$

### Example 1.2

If a function is given by

$$F(s) = Q \left[ \frac{(s+a)\sin\theta}{(s+a)^2 + \beta^2} + \frac{\beta\cos\theta}{(s+a)^2 + \beta^2} \right]$$

Show that the initial value of this function in time domain is equal to  $f(0^+) = Q\sin\theta$

### Solution

Using initial value theorem

$$f(0^+) = \lim_{s \rightarrow \infty} sF(s)$$

$$F(s) = Q \left[ \frac{\beta \cos \theta + (s+a) \sin \theta}{(s+a)^2 + \beta^2} \right]$$

$$sF(s) = Q \left[ \frac{s\beta \cos \theta + s(s+a) \sin \theta}{(s+a)^2 + \beta^2} \right]$$

Divide numerator and denominator by  $s^2$

$$sF(s) = Q \left[ \frac{\frac{\beta \cos \theta}{s} + \left(1 + \frac{a}{s}\right) \sin \theta}{\frac{(s+a)^2}{s^2} + \frac{\beta^2}{s^2}} \right]$$

$$\lim_{s \rightarrow \infty} sF(s) = Q \left[ \frac{0 + (1+0) \sin \theta}{1+0} \right]$$

$$\lim_{s \rightarrow \infty} sF(s) = Q \sin \theta$$

$$f(0^+) = Q \sin \theta$$

### Example 1.3

Find the initial value of

$$f(t) = e^{-\theta} (\sin 3\theta + \cos 5\theta)$$

### Solution

$$f(t) = e^{-\theta} \sin 3\theta + e^{-\theta} \cos 5\theta$$

$$F(s) = \frac{3}{(s+1)^2 + 3^2} + \frac{s+1}{(s+1)^2 + 5^2}$$

$$F(s) = \frac{3}{(s+1)^2 + 9} + \frac{s+1}{(s+1)^2 + 25}$$

$$sF(s) = \frac{3s}{(s+1)^2 + 9} + \frac{s^2 + s}{(s+1)^2 + 25}$$

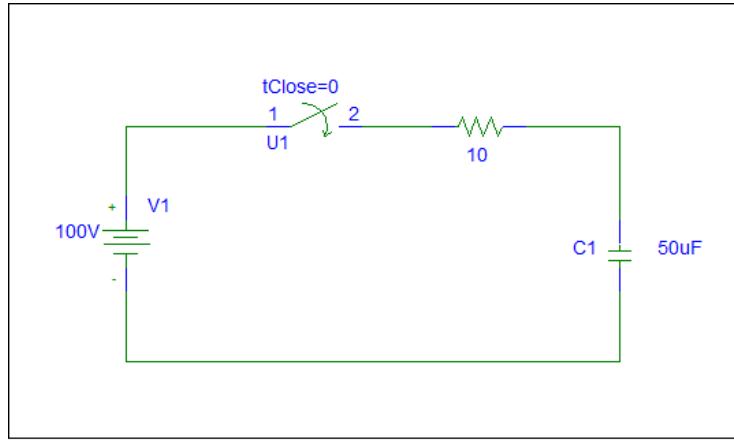
Divide numerator and denominator by  $s^2$

$$\lim_{s \rightarrow \infty} sF(s) = \frac{\frac{3}{s}}{1 + \frac{2}{s} + \frac{1}{s^2} + \frac{9}{s^2}} + \frac{\frac{1}{s} + \frac{1}{s}}{1 + \frac{2}{s} + \frac{1}{s^2} + \frac{25}{s^2}}$$

$$\lim_{s \rightarrow \infty} sF(s) = 1$$

$$f(0^+) = 1$$

### Example 1.4



In the RC network shown the capacitor has an initial charge  $q_0 = 2500\mu C$ . At  $t = 0$ , the switch is closed and a constant voltage source  $V = 100V$  is applied to the circuit. Use Laplace transforms method to find the current.

### Solution

$$Ri(t) + \frac{1}{c} \int i(t) \delta t = V$$

$$RI(s) + \frac{1}{c} \left[ \frac{I(s)}{s} + \frac{1}{s} \int i(t) \delta t_{0^+} \right] = \frac{V}{s}$$

$$RI(s) + \frac{I(s)}{sc} + \frac{1}{s} Q_0 = \frac{V}{s}$$

$$I(s) \left( R + \frac{1}{sc} \right) + \frac{Q_0}{sc} = \frac{V}{s}$$

$$I(s) \left( s + \frac{1}{Rc} \right) \frac{R}{s} + \frac{Q_0}{sc} = \frac{V}{s}$$

$$I(s) = \frac{\left( \frac{V}{s} - \frac{Q_0}{sc} \right)}{\frac{R}{s} \left( s + \frac{1}{Rc} \right)}$$

$$I(s) = \left( \frac{V}{s} - \frac{Q_0}{sc} \right) \frac{s}{R} \left( \frac{1}{s + \frac{1}{Rc}} \right)$$

$$I(s) = \left( \frac{V}{R} - \frac{V_0}{R} \right) \left( \frac{1}{s + \frac{1}{Rc}} \right)$$

$$\frac{1}{Rc} = \frac{1}{10 \times 50 \times 10^{-6}} = 2000, V_0 = \frac{Q_0}{c} = \frac{2500 \times 10^{-6}}{50 \times 10^{-6}} = 50V$$

$$I(s) = \frac{V}{R} \left( \frac{1}{s + 2000} \right) - \frac{V_0}{R} \left( \frac{1}{s + 2000} \right)$$

$$I(s) = \frac{100}{10} \left( \frac{1}{s + 2000} \right) - \frac{50}{10} \left( \frac{1}{s + 2000} \right)$$

$$I(s) = \left( \frac{10}{s + 2000} \right) - \left( \frac{5}{s + 2000} \right)$$

Transform back to time domain

$$i(t) = 10e^{-2000t} - 5e^{-2000t} A$$

### Example 1.5

A shifted unit step function is expressed as  $f(t) = u(t - a)$ . Obtain it's Laplace transform.

### Solution

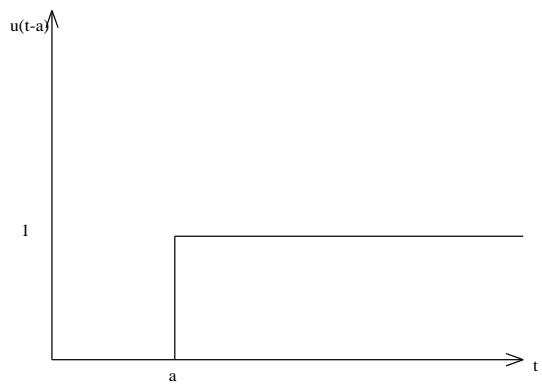
A unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



But when it is time shifted or delayed by a

$$u(t - a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



$$\begin{aligned}
 \mathcal{L}u(t-a) &= \int_a^{\infty} u(t) \cdot e^{-st} dt = \int_a^{\infty} 1 \cdot e^{-st} dt \\
 &= \left[ -\frac{e^{-st}}{s} \right]_a^{\infty} = \frac{e^{-as}}{s} \\
 \mathcal{L}u(t-a) &= e^{-as} \frac{1}{s}
 \end{aligned}$$