Inverse Laplace Transforms

Inverse Laplace by Inspection

To convert a function in s domain back to the time domain, the function must be expressed in a format that is easy to relate with the formula obtained during the transformation from time domain to s domain

Example 2.1

If $F(s) = \frac{5}{s}$ then the inverse Laplace is 5

Example 2.2

If $F(s) = \frac{9}{s^2+3^2}$

This can be written as

$F(s) = \frac{9}{s^2+3^2}$ and the inverse Laplace is $f(t) = 3sin3t$

By simply comparing $F(s)$ to the formula obtained earlier the function can be transformed to t-domain.

Partial Fraction method for Inverse Laplace Transforms

Example 2.3 (Simple pole)

If $F(s) = \frac{s+2}{s^2-5s+6}$ find $\mathcal{L}^{-1}F(s)$

Solution

The poles are s=2 and s=3

$$F(s) = \frac{s+2}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

Method 1

Residue method

$$A = \frac{(s+2)_{s=2}}{(s-2)(s-3)_{s-2}} = \frac{4}{-1}$$

$$A = -4$$

$$B = \frac{(s+2)_{s=3}}{(s-2)(s-3)_{s-3}} = \frac{5}{1} = 5$$

$$F(s) = \frac{5}{(s-3)} - \frac{4}{(s-2)}$$

$$\mathcal{L}^{-1}F(s) = f(t) = 5e^{3t} - 4e^{2t}$$

Method 2

Algebraic method

$$F(s) = \frac{s+2}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

Multiply through by $(s-2)(s-3)$ to make it linear

$$(s+2) = A(s-3) + B(s-2)$$

$$(s+2) = As - 3A + Bs - 2B$$

$$(s+2) = (A+B)s - 3A - 2B$$

Equate coefficient of s

$$1 = A + B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .1$$

Equate constant terms

$$2 = -3A - 2B \ldots\ldots\ldots\ldots . .2$$

$$A = 1 - B \ldots\ldots\ldots\ldots\ldots 3$$

Put 3 in 2

$$2 = -3(1-B) - 2B$$

$$2 = -3 + 3B - 2B$$

$$2 = -3 + B$$

$$B = 5$$

$$A = 1 - 5 = -4$$

$$F(s) = \frac{5}{(s-3)} - \frac{4}{(s-2)}$$

$$\mathcal{L}^{-1}F(s) = f(t) = 5e^{3t} - 4e^{2t}$$

Example 2.4 (repeated poles)

Find $v(t)$ if $V(s) = \frac{10s^2+4}{s(s+1)(s+2)^2}$

Solution

$$\frac{10s^2+4}{s(s+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)}$$

By residue method

$$A = sV(s)_{s=0} = \frac{10s^2+4}{s(s+1)(s+2)^2}_{s=0} = \frac{4}{4} = 1$$

$$B = (s+1)V(s)_{s=-1} = \frac{10s^2+4}{s(s+1)(s+2)^2}_{s=-1} = \frac{-14}{1} = -14$$

$$C = (s+2)^2 V(s)_{s=-2} = \frac{10s^2+4}{s(s+1)(s+2)^2}_{s=-2} = \frac{44}{(-2)(-1)} = \frac{44}{2} = 22$$

$$D = \frac{d}{ds}[(s+2)^2 V(s)]_{s=-2}$$

$$D = \frac{d}{ds}\left[\frac{10s^2+4}{s^2+s}\right]$$

$$D = \frac{(s^2+s)20s - (10s^2+4)(2s+1)}{(s^2+s)^2}$$

$$D = \frac{(4-2)(-40) - (40+4)(-3)}{4}$$

$$D = \frac{2(-40) + 44(3)}{4}$$

$$D = \frac{-80+132}{4} = \frac{52}{4} = 13$$

$$F(s) = \frac{1}{s} - \frac{-14}{s+1} + \frac{22}{(s+2)^2} + \frac{13}{s+2}$$

$$\mathcal{L}^{-1}F(s) = f(t) = 1 - 14e^{-t} + 22te^{-2t} + 13e^{-2t}$$

Heavy Side Formula

If $F(s) = \frac{M(s)}{N(s)}$

$$\mathcal{L}^{-1}F(s) = \sum_{k=1}^{n} \frac{M(a_k)}{N^{|}(a_k)} e^{akt}$$

Where a_k is number of distinct roots of $N(s)$, $N^{|}(s)$ is the differential of $N(s)$. This formula is used when the roots are simple (i.e. not repeated).

Example 2.5

Use the Heaviside formulae to solve example 2.3

Solution

$$F(s) = \frac{s+2}{s^2 - 5s + 6}$$

The poles are $s = 2$ and $s = 3$

$$N^{|}(s) = 2s - 5$$

Applying the Heaviside formulae

$$\mathcal{L}^{-1}F(s) = \frac{s+2}{2s-5}_{s=2} e^{2t} + \frac{s+2}{2s-5}_{s=3} e^{3t}$$

$$\mathcal{L}^{-1}F(s) = \frac{4}{-1} e^{2t} + \frac{5}{1} e^{3t}$$

$$\mathcal{L}^{-1}F(s) = 5e^{3t} - 4e^{2t}$$

Example 2.6

A function in s-domain is given by

$$F(s) = \frac{50}{s^2 + 2s + 2}$$

Find the inverse Laplace transform.

Solution

$$F(s) = \frac{50}{s^2 + 2s + 2}$$

This can be written as

$$F(s) = \frac{50}{s^2 + 2s + 1 + 1} = \frac{50}{(s+2)^2 + 1}$$

$$f(t) = 50e^{-t} sint$$

Example 2.7

Find the inverse Laplace of

$$F(s) = \frac{1}{s} + \frac{2}{s+1}$$

Solution

$$\mathcal{L}^{-1}F(s) = 1u(t) + 2e^{-t}$$

$$\mathcal{L}^{-1}F(s) = u(t) + 2e^{-t}$$

Example 2.8

Find the inverse Laplace of

$$F(s) = \frac{3s+1}{s+4}$$

Solution

The denominator contains a single simple pole @ $s = -4$ looking at the polynomial the numerator is bigger than the denominator. Hence, divide using long division.

$$3$$

$$\sqrt[s+4]{3s+1}$$

$$\underline{3s + 12}$$

$$-11$$

$$= 3 - \frac{11}{s+4}$$

$$\mathcal{L}^{-1}F(s) = 3\delta(t) - 11e^{-4t}$$

$\delta(t)$ is the unit impulse function

Example 2.9

Find the inverse Laplace of

$$F(s) = \frac{4}{(s+1)(s+3)}$$

Solution

$$\frac{4}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$A = \frac{4}{2} = 2$$

$$B = \frac{4}{-2} = -2$$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+3}$$

$$\mathcal{L}^{-1}F(s) = f(t) = 2e^{-t} - 2e^{-3t}$$

Example 2.10

Find the inverse Laplace of

$$F(s) = \frac{12}{(s+2)^2(s+4)}$$

Solution

$$\frac{12}{(s+2)^2(s+4)} = \frac{A}{s+4} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

$$A = \frac{12}{(s+2)^2}_{s=-4}$$

$$A = \frac{12}{4} = 3$$

$$B = \frac{12}{(s+4)}_{s=-2}$$

$$B = \frac{12}{2} = 6$$

$$C = \frac{d}{ds}\left[\frac{12}{(s+4)}\right]_{s=-2}$$

$$B = \frac{-12}{4} = -3$$

$$F(s) = \frac{3}{s+4} - \frac{6}{(s+2)^2} - \frac{3}{s+2}$$

$$\mathcal{L}^{-1}F(s) = f(t) = 3e^{-4t} - 3e^{-2t} + -6te^{-2t}$$

Example 2.11

If the Laplace transform of the voltage across a capacitor in an RC circuit is

$$V_c(s) = \frac{20 + 2s}{s(s+2)}$$

a) Find the initial capacitor voltage
b) Find the voltage of the D.C. source
c) Calculate the value of the resistor if the current at $t = 0$ is 2.42mA.
d) Draw the circuit and determine the value of the capacitor.

Solution

a using initial value theorem

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

$$V_c(0^+) = \lim_{s\to\infty} sV_c(s) = \left.\frac{20 + 2\mathrm{s}}{\mathrm{s} + 2}\right|_{s=\infty}$$

$$V_c(0^+) = \left.\frac{\frac{20}{s} + 2}{1 + \frac{2}{s}}\right|_{s=\infty}$$

$$V_c(0^+) = 2V$$

b since the capacitor will charge up to the value of the source at $t = \infty$, then using final value theorem

$$f(\infty) = \lim_{s\to 0} sF(s)$$

$$V_{source} = V_c(\infty) = \lim_{s\to 0} \frac{20 + 2s}{s + 2}$$

$$V_{source} = \frac{20}{2} = 10V$$

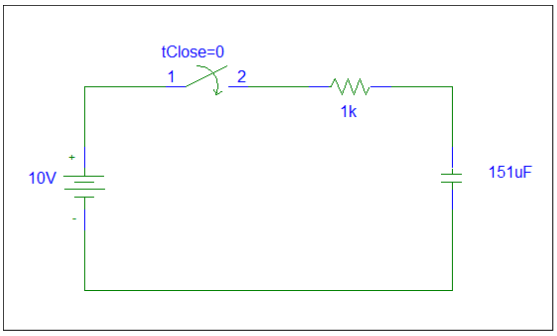
c if $i(0^+) = 2.42mA$ then

$$V_R = V_{source} - V_c(0^+)$$

$$V_R = 10 - 2 = 8V$$

$$R = \frac{V_R}{i(0^+)} = \frac{8}{2.42 \times 10^{-3}} = 3.3 \times 10^3 = 3.3K\Omega$$

d



$$\alpha = \frac{1}{RC}, c = \frac{1}{\alpha R} = \frac{1}{2 \times 3.3 \times 10^3}$$

$$c = 151.5 \times 10^{-6} F$$

$$c = 151 \mu F$$

Pole and Zero Plot

Given a network with transfer function in complex frequency expressed as

$$H(s) = \frac{P(s)}{Q(s)}$$

The roots of the denominator are called “poles”; a pole is a critical frequency at which the response of the system has infinite magnitude.

The roots of the numerator are called “zeroes”; a zero is a critical frequency at which the response has zero amplitude.

A plot of the poles and zeroes on the s-plane with $j\omega$ in rad/sec and σ in nepers is used to express and analize the behaviour of a network graphically. Poles are plotted as ⊗ while zeroes are plotted as ⊙.

Example 2.12

$$H(s) = \frac{s^2 - 16}{s^5 - 7s^4 - 30s^3}$$

Sketch the pole-zero plots for the system

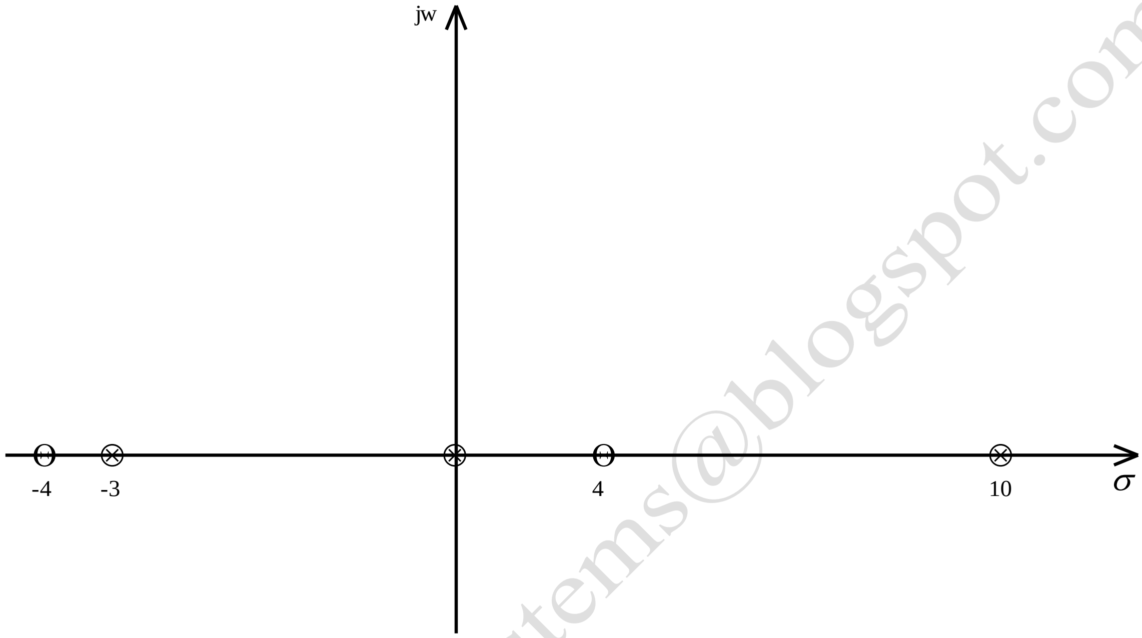
Solution

Zero; $s^2 - 16 = 0, s^2 = 16, s = \pm 4$

Poles; $s^5 - 7s^4 - 30s^3 = s^3(s^2 - 7s - 30) = 0$

$$= s^3(s - 10)(s + 3) = 0$$

Poles are $s = 0, s = -3, s = 10$



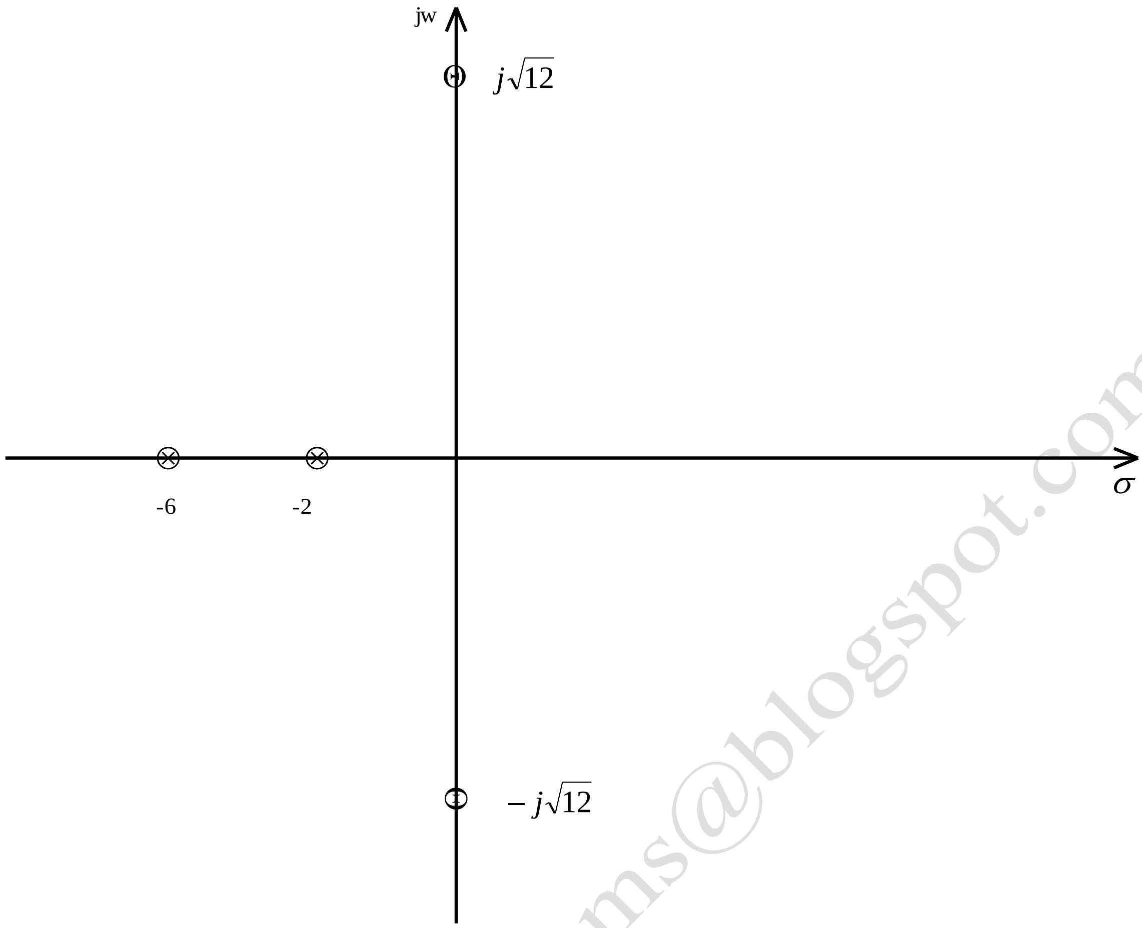
Example 2.13

$$H(s) = \frac{s^2 - 12}{s^2 + 8s + 12}$$

Sketch the pole-zero plots for the system

Solution

Zero; $s^2 + 12 = 0, s^2 = -12, s = \sqrt{-12} = \pm j\sqrt{12}$
Pole; $s^2 + 8s + 12 = 0, (s + 2)(s + 6) = 0, s = -2, s = -6$



Example 2.14

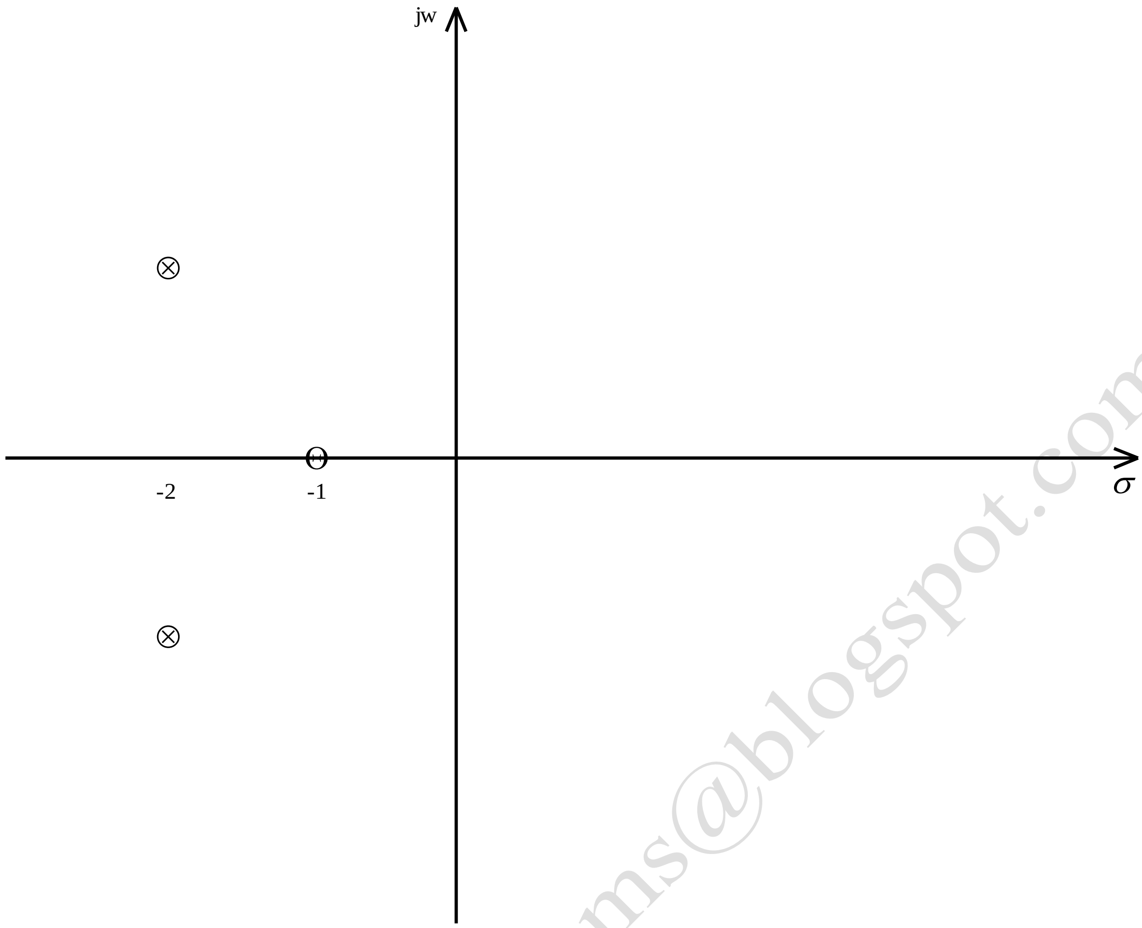
$$H(s) = \frac{5(s+1)}{s^2 + 4s + 5}$$

Sketch the pole-zero plots for the system

Solution

Poles; $s^2 + 4s + 5 = 0, s = -2 + j, s = -2 - j$

Zero; $s + 1 = 0, s = -1$


jw
σ
-2
-1