

### 3-Transmission Lines

A transmission line is a guided structure used to efficiently transmit power or information from the source to the load. Transmission lines are used in power distribution (at low frequencies) and in communications (at high frequencies).

Transmission lines basically consist of two or more parallel conductors used to connect a source to a load, example of transmission lines are coaxial cable, two –wire line, parallel plate and microstrip line.

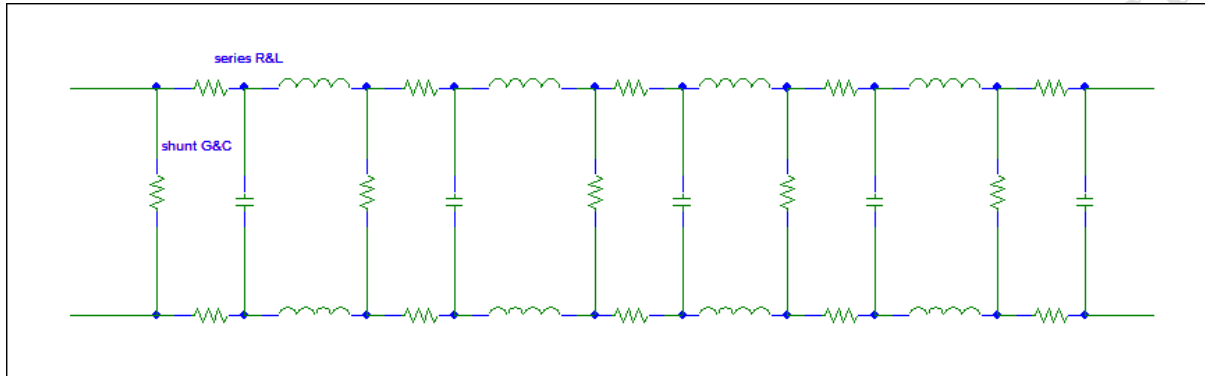


Figure 3.1 Distributed parameters of a two conductor transmission line

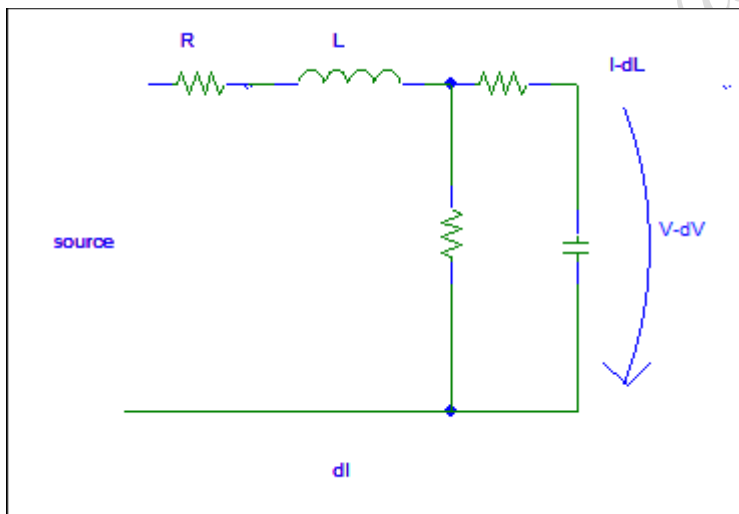


Figure 3.1 Parameters of a short section  $\delta l$  of a transmission line

Taking a small portion, an equivalent circuit can be derived as shown above the voltage decreases as the length of the line increases

$$\frac{\delta v}{\delta l} = -(R + j\omega L)I \quad 3.1$$

The current decreases as the length increases along the line

$$\frac{\delta I}{\delta l} = -(G + j\omega C)V \quad 3.2$$

$$\frac{\delta^2 v}{\delta l^2} = -(R + j\omega L) \frac{\delta I}{\delta l} \quad 3.3$$

Put 3.2 in 3.3

$$\frac{\delta^2 v}{\delta l^2} = (R + j\omega L)(G + j\omega C)V = \gamma^2 V \quad 3.4$$

Where  $\gamma$  is the propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad 3.5$$

And  $\gamma = \alpha + j\beta$ , where  $\alpha$  is the attenuation constant in Nepers/m and  $\beta$  is the phase constant in radians/m.

The voltage  $V$  consist of the forward travelling wave and the backward travelling wave represented as

$$V = Ae^{-\gamma l} + Be^{\gamma l} \quad 3.6$$

$$\frac{\delta v}{\delta l} = -\gamma Ae^{-\gamma l} + \gamma Be^{\gamma l} \quad 3.7$$

$$\frac{\delta v}{\delta l} = -\gamma(Ae^{-\gamma l} - Be^{\gamma l}) \quad 3.8$$

Equate 3.8 to 3.3

$$\frac{\delta v}{\delta l} = -\gamma(Ae^{-\gamma l} - Be^{\gamma l}) = -(R + j\omega L)I \quad 3.9$$

$$I = \frac{\gamma}{R + j\omega L} (Ae^{-\gamma l} - Be^{\gamma l}) \quad 3.10$$

$$\frac{\gamma}{R + j\omega L} = \sqrt{\frac{(R + j\omega L)(G + j\omega C)}{(R + j\omega L)^2}} \quad 3.11$$

$$I = \sqrt{\frac{G + j\omega C}{R + j\omega L}} (Ae^{-\gamma l} - Be^{\gamma l}) = \frac{1}{Z_0} (Ae^{-\gamma l} - Be^{\gamma l}) \quad 3.12$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

With the load as reference  $l = 0$

Equation 3.6 becomes

$$V = A + B = V_s \quad 3.13$$

And equation 3.12

$$V = \frac{A-B}{Z_0} = I_s \quad 3.14$$



Looking from the generator side  $Z_L$  is seen

$$Z_L = \frac{V_L}{I_L} = \frac{Ae^{-\gamma l} + Be^{\gamma l}}{\frac{1}{Z_0}(Ae^{-\gamma l} - Be^{\gamma l})} \quad 3.15$$

$$\cosh \gamma t = \frac{e^{\gamma t} + e^{-\gamma t}}{2}$$

$$\sinh \gamma t = \frac{e^{\gamma t} - e^{-\gamma t}}{2}$$

$$e^{\gamma t} = \cosh \gamma t + \sinh \gamma t$$

$$e^{-\gamma t} = \cosh \gamma t - \sinh \gamma t$$

Substituting these in 3.15

$$Z_L = \frac{A(\cosh \gamma t - \sinh \gamma t) + B(\cosh \gamma t + \sinh \gamma t)}{\frac{A}{Z_0}(\cosh \gamma t - \sinh \gamma t) - \frac{B}{Z_0}(\cosh \gamma t + \sinh \gamma t)}$$

$$Z_L = \frac{(A+B)\cosh \gamma t - (A-B)\sinh \gamma t}{\frac{A-B}{Z_0}\cosh \gamma t - \frac{A+B}{Z_0}\sinh \gamma t} \quad 3.16$$

Substituting 3.13 and 3.14 in 3.16

$$Z_L = \frac{V_s \cosh \gamma t - I_s Z_0 \sinh \gamma t}{I_s \cosh \gamma t - \frac{V_s}{Z_0} \sinh \gamma t} \quad 3.17$$

$$\frac{V_s}{I_s} = Z_s$$

Divide numerator and denominator by  $V_s$

$$Z_L = \frac{\cosh \gamma t - \frac{Z_0}{Z_s} \sinh \gamma t}{\frac{1}{Z_s} \cosh \gamma t - \frac{1}{Z_0} \sinh \gamma t} \quad 3.18$$

Multiply numerator and denominator of 3.18 by  $Z_s$

$$Z_L = \frac{Z_s \cosh \gamma l - Z_0 \sinh \gamma l}{\cosh \gamma l - \frac{Z_s}{Z_0} \sinh \gamma l} \quad 3.19$$

Multiply numerator and denominator by  $Z_0$

$$Z_L = \frac{Z_0 (Z_s \cosh \gamma l - Z_0 \sinh \gamma l)}{Z_0 \cosh \gamma l - Z_s \sinh \gamma l} \quad 3.20$$

To find the input impedance of the network, make  $Z_s$  the subject

From 3.20

$$\begin{aligned} Z_L Z_0 \cosh \gamma l - Z_L Z_s \sinh \gamma l &= Z_0 Z_s \cosh \gamma l - Z_s^2 \sinh \gamma l \\ Z_0 (Z_L \cosh \gamma l + Z_0 \sinh \gamma l) &= Z_s (Z_0 \cosh \gamma l - Z_L \sinh \gamma l) \\ Z_s = Z_{in} &= \frac{Z_0 (Z_L \cosh \gamma l + Z_0 \sinh \gamma l)}{Z_0 \cosh \gamma l - Z_L \sinh \gamma l} \end{aligned} \quad 3.21$$

Equation 3.21 is the general formula for input impedance of transmission lines.

The wavelength of the wave is

$$\lambda = \frac{2\pi}{\beta}$$

The wave velocity is

$$u = \frac{\omega}{\beta} = f\lambda$$

### Lossless Transmission Line

$$R = 0 = G$$

$$\gamma = \alpha + j\beta$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma = \sqrt{j\omega^2 LC} = j\omega\sqrt{LC} = \alpha + j\beta$$

This means for a lossless line

$$\alpha = 0, \text{ and } \beta = \omega\sqrt{LC}$$

$$\lambda = \frac{2\pi}{\beta}, u = f\lambda$$

$$u = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

But

$$R = 0 = G$$

$$\therefore Z_0 = \sqrt{\frac{L}{C}} = R_0 + jX_0$$

$$R_0 = \sqrt{\frac{L}{C}}, X_0 = 0$$

### Distortionless Line

$$\frac{R}{L} = \frac{G}{C} \text{ or } \frac{R}{G} = \frac{L}{C}$$

$$Y = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$Y = \sqrt{\left(1 + j\omega \frac{L}{R}\right) \left(1 + j\omega \frac{C}{G}\right) RG}$$

Since

$$\frac{L}{R} = \frac{C}{G}$$

Then

$$Y = \sqrt{RG} \left(1 + j\omega \frac{L}{R}\right) = \alpha + j\beta$$

$$Y = \sqrt{RG} + j\omega \frac{L}{R} \sqrt{RG}$$

$$Y = \sqrt{RG} + j\omega \sqrt{\frac{L^2}{R^2} RG}$$

$$Y = \sqrt{RG} + j\omega \sqrt{L^2 \cdot \frac{G}{R}}$$

But

$$\frac{G}{R} = \frac{C}{L}$$

$$Y = \sqrt{RG} + j\omega \sqrt{L^2 \cdot \frac{C}{L}}$$

$$Y = \sqrt{RG} + j\omega\sqrt{LC}$$

For a distortionless line

$$\alpha = \sqrt{RG} \text{ and } \beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{\left(1 + j\omega \frac{L}{R}\right) R}{\left(1 + j\omega \frac{C}{G}\right) G}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

$$\lambda = \frac{2\pi}{\beta}, u = f\lambda$$

$$u = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}$$

### Example 3.1

An air line has characteristics impedance of  $50\Omega$  and phase constant of  $5\text{rad/m}$ , at frequency of  $200\text{MHz}$ . Calculate the inductance per meter and the capacitance per meter of the line.

### Solution

An air line can be regarded as a lossless line since  $\sigma = 0$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\beta = \omega\sqrt{LC}$$

$$\frac{Z_0}{\beta} = \sqrt{\frac{L}{C}} \div \omega\sqrt{LC} = \sqrt{\frac{L}{C}} \times \frac{1}{\omega\sqrt{LC}}$$

$$\frac{Z_0}{\beta} = \frac{1}{\omega} \cdot \sqrt{\frac{L}{C}} \times \frac{1}{\sqrt{LC}} = \frac{1}{\omega} \sqrt{\frac{1}{C^2}} = \frac{1}{\omega C}$$

$$\frac{50}{5} = \frac{1}{\omega C} = 10$$

$$C = \frac{1}{10\omega} = \frac{1}{2 \times \pi \times 200 \times 10^6 \times 10} = 79.57 \text{ pF/m}$$

$$\beta = \omega \sqrt{LC}$$

$$\frac{\beta}{\omega} = \sqrt{LC} = \frac{5}{2 \times \pi \times 200 \times 10^6} = 3.97 \times 10^{-9}$$

$$LC = (3.97 \times 10^{-9})^2$$

$$L = \frac{1.5831 \times 10^{-17}}{79.57 \times 10^{-12}} = 198.93 \times 10^{-9} \text{ H} = 198.93 \text{ nH/m}$$

### Example 3.2

A distortionless line has  $Z_0 = 60\Omega$ ,  $\alpha = 20 \text{ mNp/m}$ ,  $u = 0.6c$ , where  $c$  is the speed of light in a vacuum. Find  $R$ ,  $L$ ,  $G$ ,  $C$  and  $\lambda$  at  $100 \text{ MHz}$ .

### Solution

$$\frac{G}{R} = \frac{C}{L}$$

$$Z_0 = 60\Omega, \alpha = 20 \text{ mNp/m}, u = 0.6c, f = 100 \text{ MHz}$$

$$\beta = \omega \sqrt{LC}$$

$$\alpha = \sqrt{RG}$$

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}$$

$$\gamma = \sqrt{RG} + j\omega \sqrt{LC}$$

$$\alpha = 20 \times 10^{-3} = \sqrt{RG}$$

$$Z_0 = 60 = \sqrt{\frac{R}{G}}$$

$$\alpha Z_0 = 20 \times 10^{-3} \times 60 = \sqrt{\frac{R}{G}} \times \sqrt{RG} = R$$

$$R = 1.2 \Omega/\text{m}$$

$$RG = \alpha^2$$

$$G = \frac{\alpha^2}{R} = \frac{(20 \times 10^{-3})^2}{1.2} = 333 \mu S/m$$

$$\beta = \frac{\omega}{u} = \frac{2 \times \pi \times 100 \times 10^6}{0.6 \times 3 \times 10^8} = 3.49 \text{ rad/m}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\beta = \omega \sqrt{LC}$$

$$Z_0 \beta = \omega \sqrt{\frac{L}{C}} \cdot LC = \omega L$$

$$L = \frac{Z_0 \beta}{\omega}$$

$$L = \frac{60 \times 3.49}{2 \times \pi \times 100 \times 10^6}$$

$$L = \frac{209.4}{638318530.7} = 333 \times 10^{-9} = 333 \text{ nH/m}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

$$C = \frac{L}{Z_0^2} = \frac{333 \times 10^{-9}}{60^2} = 92.5 \times 10^{-12} \text{ F/m} = 92.5 \text{ pF/m}$$

### Reflection Coefficient and Standing Wave Ratio

Since

$$V(l) = Ae^{-\gamma l} + Be^{\gamma l}$$

$$V(l) = A(e^{-\gamma l} + \rho_L e^{\gamma l})$$

A is the forward travelling voltage  $V_0^+$

A is the backward travelling voltage  $V_0^-$

$$\rho_L = \frac{B}{A} = \frac{V_0^-}{V_0^+}$$

$\rho_L$  is the voltage reflection coefficient (at the load) defined as the ratio of the reflected wave to the incident wave.

From equation 3.12

$$I(l) = \frac{A}{Z_0} e^{-\gamma l} - \frac{B}{Z_0} e^{\gamma l}$$

$$I(l) = \frac{A}{Z_0} (e^{-\gamma l} - \rho_L e^{\gamma l}), \quad \frac{A}{Z_0} = I_0^+, \quad -\frac{B}{Z_0} = I_0^-$$

$$\frac{I_0^-}{I_0^+} = \rho_L$$

Generally the voltage reflection coefficient at any point on the line can be defined as the ratio of the magnitude of the reflected voltage wave to that of the incident wave.

$$\rho(l) = \frac{V_0^- e^{\gamma l}}{V_0^+ e^{-\gamma l}} = \frac{V_0^-}{V_0^+} e^{2\gamma l} = \rho_L e^{2\gamma l}$$

At the load  $l = 0$

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ (1 + \rho_L)}{\frac{V_0^+}{Z_0} (1 + \rho_L)} = Z_0 \left( \frac{1 + \rho_L}{1 - \rho_L} \right)$$

$$Z_L = Z_0 \left( \frac{1 + \rho_L}{1 - \rho_L} \right)$$

$$Z_L - Z_0 \rho_L = Z_0 + Z_0 \rho_L$$

$$Z_L - Z_0 = (Z_L + Z_0) \rho_L$$

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$|\rho_L| = \frac{V_{max} - V_{min}}{V_{max} + V_{min}}$$

$$|\rho_L| = \frac{\frac{V_{max}}{V_{min}} - 1}{\frac{V_{max}}{V_{min}} + 1} = \frac{s - 1}{s + 1}$$

$$s - 1 = |\rho_L|s + |\rho_L|$$

$$s - |\rho_L|s = 1 + |\rho_L|$$

$$s(1 - |\rho_L|) = 1 + |\rho_L|$$

$$s = \frac{1 + |\rho_L|}{1 - |\rho_L|}$$

### Shorted Line ( $Z_L = 0$ )

$$Z_{sc} = \frac{Z_0(Z_L \cosh \gamma l + Z_0 \sinh \gamma l)}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l} = \frac{Z_0(Z_L + Z_0 \tanh \gamma l)}{Z_0 + Z_L \tanh \gamma l}$$

$$Z_{sc} = \frac{Z_0(Z_0 \sinh \gamma l)}{Z_0 \cosh \gamma l} = Z_0 \tanh \gamma l$$

$$Z_{sc} = Z_0 \tanh \gamma l$$

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$$

$$s = \frac{1 + |\rho_L|}{1 - |\rho_L|} = \infty$$

### Open Circuited Line ( $Z_L = \infty$ )

$$Z_{oc} = \frac{Z_0(\infty \cosh \gamma l + Z_0 \sinh \gamma l)}{Z_0 \cosh \gamma l + \infty \sinh \gamma l}$$

$$Z_{oc} = \frac{Z_0 \infty \cosh \gamma l}{\infty \sinh \gamma l} = Z_0 \coth \gamma l$$

$$\rho_L = 1$$

$$s = \infty$$

Since  $\rho_L = 1$  it means  $V_0^+ = V_0^-$

Observe that  $Z_{sc} \times Z_{oc} = Z_0^2$

**Matched Line ( $Z_L = Z_0$ )**

$$Z_{in} = \frac{Z_0(Z_0 \cosh \gamma l + Z_L \sinh \gamma l)}{Z_0 \cosh \gamma l + Z_L \sinh \gamma l}$$

$$Z_{in} = Z_0$$

$$\rho_L = 0$$

$$s = 1$$

There is no reflection, the incident power is fully absorbed by the load, and so maximum power transfer is possible when a transmission line is matched to load.

## Use of Smith Chart

### Determination of reflection coefficient $\rho_L$

#### Example 3.3

If  $Z_0 = 80\Omega$  and  $Z_L = 35 + j50\Omega$ . Use the Smith Chart provided to calculate the reflection coefficient of the line. Indicate OP and OQ on the chart.

#### Solution

To confirm the result to be obtained from the Smith chart, calculate the value of  $\rho_L$  using the formula  $\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$\rho_L = \frac{35 + j50 - 80}{35 + j50 + 80} = \frac{-45 + j50}{115 + j50}$$
$$\rho_L = 0.5364 \angle 108.4^\circ$$

Using the Smith chart

Step 1. Obtain the normalized impedance

$$Z_{Ln} = \frac{Z_L}{Z_0} = \frac{35 + j50}{80} = 0.4375 + j0.625$$

Step 2. Locate the real value of the normalised impedance on the chart using the scale on the horizontal line at the centre and identify the circle it belongs.

Step 3. Locate the arc with value equal to the imaginary part of the normalized impedance

Step 4. Locate the point where the circle in step 2 crosses the arc in step 3, mark this as point P.

Step 5. The origin 'O' is at the centre of the chart at '1.0' and point 'Q' is located at the edge of the chart where a straight line OP passes through the '0.0' circle.

Step 6. Measure OP and OQ

$$\rho_L = \frac{OP}{OQ} = \frac{4.2}{7.8} = 0.538461$$

The angle is shown on the edge where line OP passes through.

Here it is  $108^\circ$

$$\therefore \rho_L = 0.5384 \angle 108^\circ$$

### Example 3.3

If  $Z_0 = 30\Omega$  and  $Z_L = 42 + j24\Omega$ . Use the Smith Chart to find  $\rho_L$ .

#### Solution

$$Z_{Ln} = \frac{Z_L}{Z_0} = \frac{42 + j24}{30} = 1.4 + j0.8$$

$$\rho_L = \frac{2.75}{7.8} = 0.3525 \angle -45^\circ$$

#### Determination of Standing wave ratio using the Smith chart

To obtain the standing wave ratio, draw a circle with radius OP and centre at 0. Locate point S where the S circle meets the  $\rho_{Lr}$  axis. The value of r at this point is s

$$s = r \text{ for } (r \geq 1)$$

Do this for the previous examples and confirm answer using  $s = \frac{1+|\rho_L|}{1-|\rho_L|}$

#### Determination of input impedance

$\lambda$  distance on the line corresponds to a movement of  $720^\circ$  on the chart.

$\frac{\lambda}{2}$  distance on the line corresponds to a movement of  $360^\circ$  on the chart.

- Step 1. Calculate the wavelength using the information given  $\lambda = \frac{u}{f}$
- Step 2. Determine how many wavelengths the length of the line corresponds to, then multiply by  $720^\circ$
- Step 3. Rotate along the line (clockwise i.e. towards generator) mark the point corresponding to the angle obtained.
- Step 4. Find the value of the normalized impedance at that point.
- Step 5. Finally multiply by  $Z_0$  to obtain the actual value of the input impedance  $Z_{in}$ .

**Example 3.4**

A 30m long lossless transmission line with  $Z_0 = 50\Omega$  operating at 2Mhz is terminated with a load  $Z_L = 60 + j40\Omega$ . If  $u = 0.6c$  on the line find

- The reflection coefficient  $\rho_L$ .
- The standing wave ratio.
- The input impedance.

**Solution**

$$a = 0.3523\angle 56^\circ$$

$$b = 2.1$$

$$c = (240^\circ) \quad Z_{in} = 50(0.47 + j0.035) = 23.5 + j1.75\Omega$$